Linear State-Space Control Systems

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Course Schedule

Session Topic
1. State space models of linear systems
2. Solution to State equations, canonical forms
3. Controllability and observability
4. Stability and dynamic response
5. Controller design via pole placement
6. Controllers for disturbance and tracking systems
7. Observer based compensator design
8. Linear quadratic optimal control
9. Kalman filters and stochastic control
10. LM in control design
Controllability and Observability
Controllability and Observability

• Controllability

A system is controllable if and only if it is possible by means of an input $u(t)$ to transfer the system from any initial state $x(t) = x_t$ to any final state $x(T') = x_T$ in finite time $T - t \geq 0$

• Observability

An unforced system is observable if and only if it is possible to determine any state $x(t) = x_t$ by observing the output $y(\tau)$ for a finite time $t \leq \tau \leq T$

• Note, controllability and observability concepts are specific to the state-space system description
Uncontrollable Systems

- Systems modeled with redundant state variables are uncontrollable

Let \( \dot{x} = Ax + Bu \)

\( \dot{z} = FAx + FBu \)

Where \( F \) is a \( n \times k \) matrix

Then

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
FA & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
B \\
FB
\end{bmatrix} u
\]

For \( T = \begin{bmatrix}
I_k & 0 \\
-F & I_n
\end{bmatrix} \), \( T^{-1} = \begin{bmatrix}
I_k & 0 \\
F & I_n
\end{bmatrix} \)

\( TAT^{-1} = T = \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}, \quad TB = \begin{bmatrix}
B \\
0
\end{bmatrix} \)
Unobservable Systems

- Let $\ddot{x} + 3\dot{x} + 2x = u$
- $y = x + \dot{x}$
- $H(s) = \frac{1}{s+2}$
Controllability

- Consider a general state space system model
  \[ \dot{x} = Ax + Bu \]
  \[ y = Cx + Du \]

Let \( \Phi(t, \tau) \) denote the state transition matrix, then

\[ x(T) = x_T = \Phi(T, t)x_t + \int_t^T \Phi(T, \tau)B(\tau)u(\tau)d\tau \]

\[ x_T - \Phi(T, t)x_t = \int_t^T \Phi(T, \tau)B(\tau)u(\tau)d\tau \]

Define \( P(T, t) = \int_t^T \Phi(T, \tau)B(\tau)B'(\tau)\Phi'(T, \tau) \ d\tau \)

Then, for

\[ u(\tau) = B'(\tau)\Phi'(T, \tau)P^{-1}(T, t)[x_T - \Phi(T, t)x_t], \ t \leq \tau \leq T \]

\[ x_T = \Phi(T, t)x_t + P(T, t)P^{-1}(T, t)[x_T - \Phi(T, t)x_t] = x_T \]
Controllability Theorem

• Controllability Theorem:
  A system is controllable if and only if $P(T, t)$ is nonsingular for some $T > t$

• Proof:
  Assume $P(T, t)$ is singular for all $T > t$ then for some nonzero $v$

  $v'P(T, t)v = 0 \rightarrow B'(\tau)\Phi'(T, \tau)v = 0$

  i.e. state $v$ is not achievable from the origin
LTI Systems

• For an LTI system,
  \[ \Phi(T - \tau) = e^{A(T-\tau)} \]
  \[ P(T') = \int_0^T e^{A\tau} BB' e^{A'\tau} \, d\tau \]

• Algebraic Controllability Theorem
  The LTI system \( \dot{x} = Ax + Bu \) is controllable if and only if rank \( r(Q) = n \) for controllability test matrix \( Q = [B \ AB \ldots \ A^{n-1}B] \)

• Proof: By Cayley-Hamilton theorem

  \[ e^{At}B = [I f_1(t) + Af_2(t) + \cdots + A^{n-1}f_n(t)]B = Q \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} \]

  \[ P(T) = Q \int_0^T F(t)F'(t)dt Q' = QGQ' \]
  If \( r(Q) < n \), then \( P(T') \) is singular
LTI Systems

• Conversely, assume that $P(T)$ is singular,

then $v'P(T,t)v = 0$, i.e., $z(t) = B'e^{A't}v = 0$ for $0 \leq t \leq T$

$\ddot{z}(t) = B'A'e^{A't}v = 0, \quad \dddot{z}(t) = B'(A')^{2}e^{A't}v = 0, \ldots, \quad z^{(n-1)}(t) = B'(A')^{n-1}e^{A't}v = 0;$

Or,

$$
\begin{bmatrix}
B' \\
B'A' \\
\vdots \\
B'(A')^{n-1}
\end{bmatrix}e^{A't}v = 0
$$

Let $e^{A't}v = \begin{bmatrix}
\alpha_1(t) \\
\alpha_2(t) \\
\vdots \\
\alpha_n(t)
\end{bmatrix}$, $Q' = [q_1 \ q_2 \ \ldots \ q_n]$

Then, $\alpha_1(t)q_1 + \alpha_2(t)q_2 + \ldots + \alpha_n(t)q_n = 0$, i.e., $r(Q) < n$
Observability

• Consider system output

\[ y(\tau) = C(\tau)\Phi(\tau, t)x_t, \quad \tau > t \]

Define

\[ M(T, t) = \int_t^T \Phi'(\tau, t)C'(\tau)C(\tau)\Phi(\tau, t)d\tau \]

If \( M(T, t) \) is nonsingular, then \( x_t \) can be solved as

\[ x_t = M^{-1}(T, t) \int_t^T \Phi'(\tau, t)C'(\tau)y(\tau)d\tau \]
Observability Theorem

- Observability Theorem:
  A system is observable if and only if $M(T, t)$ is nonsingular for some $T > t$

- Proof:
  Assume $M(T, t)$ is singular for all $T > t$ then for some nonzero $v$
  \[ w'M(T, t)w = 0 \]
  \[ C(\tau)\Phi(\tau, t)w = 0 \]
  i.e. state $w$ cannot be determined by observing the output
LTI Systems

• For an LTI system,
  \[ \Phi(\tau - t) = e^{A(\tau - t)} \]
  \[ M(T) = \int_0^T e^{A'\tau} C' C e^{A\tau} \, d\tau \]

• Algebraic Observability Theorem
  
  The LTI system \( \dot{x} = Ax, \, y = Cx \) is observable if and only if rank \( r(N) = n \) for observability test matrix \( N = [C' \quad A' C' \quad \ldots \quad (A')^{n-1} C'] \)

• Proof:
  
  Similar to the proof for controllability
Linear Transformation

- Controllability and observability are invariant under linear transformation of state variables

Let $\bar{x} = Tx$,

Then, $\bar{A} = TA T^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$

Then $\bar{Q} = TQ$, $r(\bar{Q}) = r(Q)$

And $\bar{N} = T^{-T} N$, $r(\bar{N}) = r(N)$
PBH Tests

• PBH Test of Controllability
  Pair \((A, B)\) is controllable if and only if for \(\lambda \in \sigma(A)\)
  \[
  \text{rank } [A - \lambda I \ B] = n
  \]

• PBH Test for Observability
  Pair \((A, C)\) is observable if and only if for \(\lambda \in \sigma(A)\)
  \[
  \text{rank } [A' - \lambda I \ C'] = n
  \]
Controllability Invariance under State Feedback

- Consider a general state space system model
  \[
  \dot{x} = Ax + Bu
  \]
  \[
  y = Cx + Du
  \]
- Let \( u = -Gx + v \)
  \[
  \dot{x} = (A - BG)x + Bv
  \]
- Theorem: Pair \((A, B)\) is controllable if and only if \((A - BG, B)\) is controllable for all \(G\)
Duality Theorem

• Pair \((A, B)\) is controllable if and only if pair \((A', B')\) observable
• Pair \((A, C)\) is observable if and only if pair \((A', C')\) is controllable
Controllable Subspace

• Assume \( r(Q) = k \leq n \),
  
  and consider \( \bar{x} = T^{-1}x \),
  
  \[
  T = [t_1 \ t_2 \ ... \ t_k \ t_{k+1} \ ... \ t_n]
  \]
  
  where \( \{t_1 \ t_2 \ ... \ t_k\} \) include linearly independent columns of \( Q \) and \( \{t_{k+1} \ ... \ t_n\} \perp Q \), then

  \[
  \bar{A} = T^{-1}AT = \begin{bmatrix}
  \bar{A}_1 & \bar{A}_{12} \\
  0 & \bar{A}_2
  \end{bmatrix}, \quad \bar{B} = \begin{bmatrix}
  \bar{B}_1 \\
  0
  \end{bmatrix}
  \]

  where \((\bar{A}_1, \bar{B}_1)\) is the controllable subsystem and \(\bar{A}_2\) is uncontrollable
Observable Subspace

- Assume that \( r(N) = k \leq n \), and consider \( \bar{x} = T^{-1}x \),

\[
T = [t_1 \ t_2 \ldots t_k \ t_{k+1} \ldots t_n]
\]

where \( \{t_1 \ t_2 \ldots t_k\} \) include linearly independent columns of \( N \) and \( \{t_{k+1} \ldots t_n\} \perp N \), then

\[
\bar{A} = T^{-1}AT = \begin{bmatrix}
\bar{A}_1 & 0 \\
\bar{A}_{21} & \bar{A}_2
\end{bmatrix}, \quad 
\bar{B} = \begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix}, \quad 
\bar{C} = [\bar{C}_1 \ 0]
\]

where \((\bar{A}_1, \bar{C}_1)\) is the observable subsystem and \(\bar{A}_2\) is unobservable
Kalman Decomposition

• Any LTI system can be transformed using:

\[ \vec{x} = T \vec{x}, \quad T^{-1} = [t_1 \; t_2 \; t_3 \; t_4] \]

into the following form

\[
\begin{bmatrix}
A_{11} & 0 & A_{13} & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & A_{33} & 0 \\
0 & 0 & A_{43} & A_{44}
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{\vec{x}}
\end{bmatrix}
+ \begin{bmatrix}
B_1 \\
B_2 \\
0 \\
0
\end{bmatrix} u
\]

\[ y = [C_1 \; 0 \; C_2 \; 0] \]

where \((A_{11}, B_1, C_1)\) is both controllable and observable, \((A_{22}, B_2)\) is controllable, and \((A_{33}, C_2)\) is observable and \((A_{44})\) is neither.